Asymmetric Statistical Errors

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Likelihood Functions

For simple results (large N) the errors are symmetrical and the likelihood is parabolic. No problem.

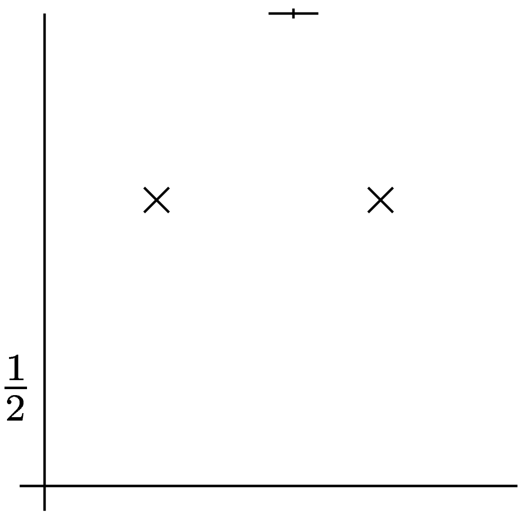
For complicated results the full likelihood function is given. No problem.

In between there are many results where the only information on the likelihood is the peak location $\hat{a}$ and the points $\hat{a} + \sigma_+$, $\hat{a} - \sigma_-$, at which the log likelihood falls by $\frac{1}{2}$

Need the likelihood (explicitly or implicitly) for combining results (just add) and combining errors (profiling).

Will (i) examine some parametrisations and (ii) apply to some toy experiments

Drawing a curved line through 3 points



Parametrise $F(a) = lnL(\vec{x}; a)$ given

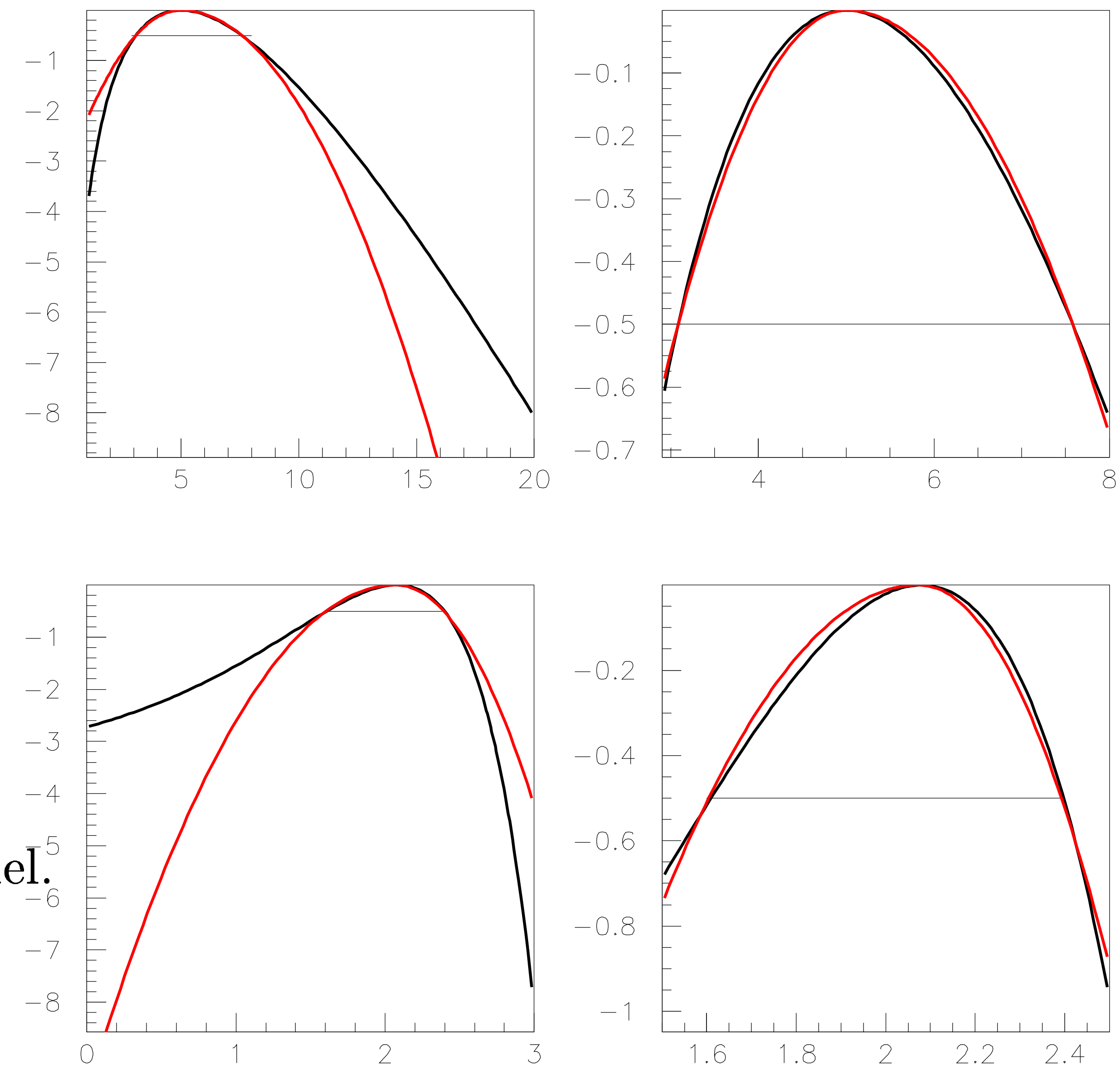
$F'(\hat{a}) = 0 \qquad F(\hat{a} + \sigma_+) = F(\hat{a}) - \frac{1}{2} \qquad F(\hat{a} - \sigma_-) = F(\hat{a}) - \frac{1}{2}$

3 parameters, (location, scale, and skew).
No manifestly right method. Seek something sensible and easy to use

Test case 1: Poisson measurement with $n = 5$. Know $L(5; a) = e^{-a}a^5/5!$. The peak is at $\hat{a} = 5$ and $\sigma_+ = 2.581$, $\sigma_- = 1.916$

Test case 2: The log of a Gaussian mean 8 and standard deviation 3. $L(8; a) = \frac{1}{3\sqrt{2\pi}} exp(\frac{(e^a - 8)^2}{18})$ and
$\hat{a} = ln(8) = 2.0744, \; \sigma_+ = ln(11) - ln(8) = 0.3184, \; \sigma_- = ln(8) - ln(5) = 0.4700$

1: Traditional Method



Split Gaussian with σ_+ above, σ_- below.

Top curves show Poisson $n = 5$.
Bottom curves show log Gaussian.

Horizontal line shows $[-\sigma_-, \sigma_+]$

Black is true likelihood. Red is model.

Right shows central region in detail.

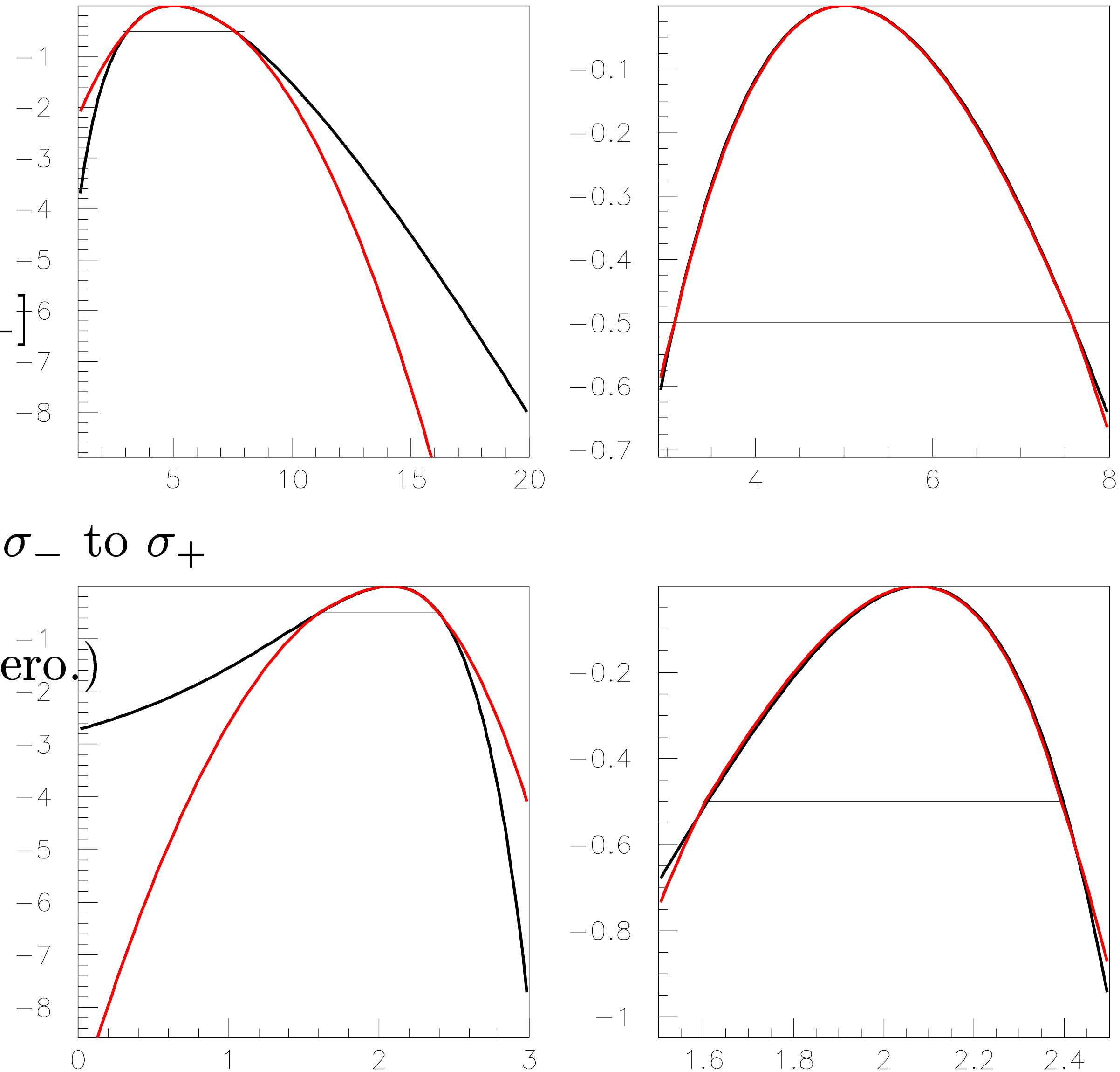
Agreement fair in central region, not good outside.

2: PDG method

Same as traditional outside $[-\sigma_-, \sigma_+]$

Within, use $-\frac{1}{2}\left(\frac{a}{\sigma_0+\sigma' a}\right)^2$
where σ interpolates smoothly from σ_- to σ_+

(Here and later formulæ take peak zero.)

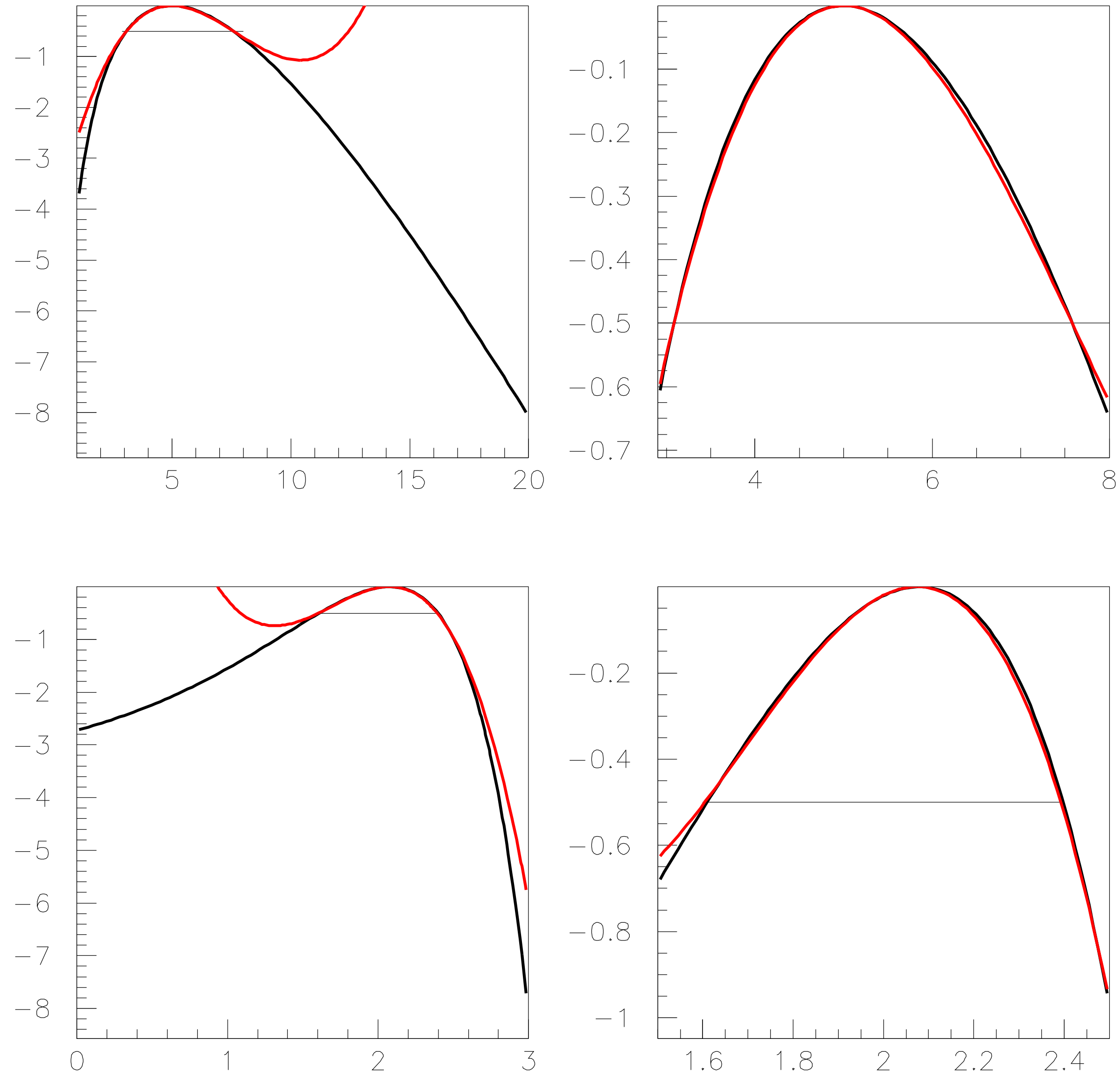


Agreement good in central region, again not good outside.

3: Cubic

Idea:
to model a curve which is nearly a parabola, add a small cubic term.

Small terms don't stay small!



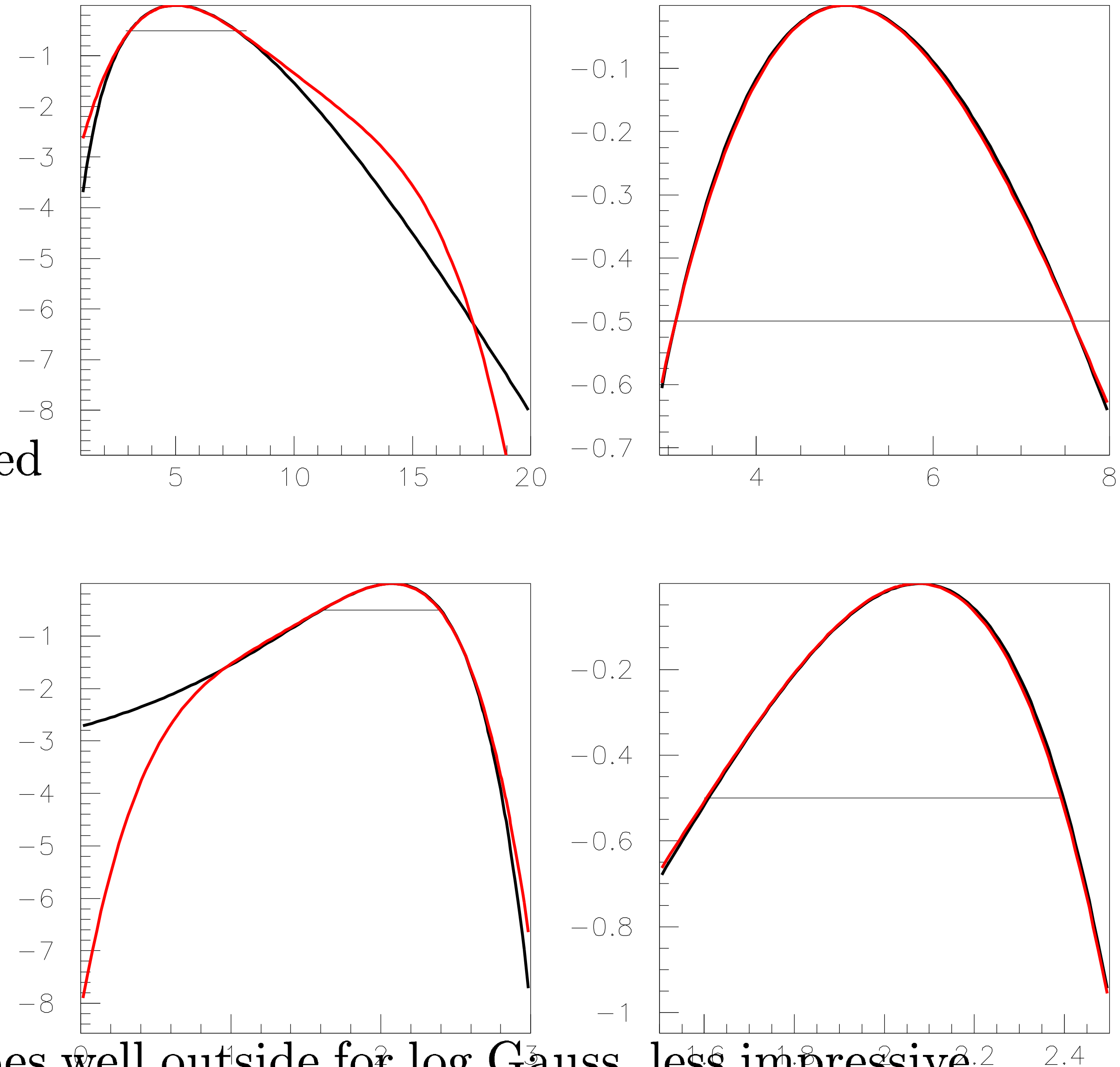
Agreement good in central region, awful very soon once outside.

4: Restricted quartic

$$lnL = -\left(\frac{\alpha^2 a^2}{2} + \frac{\alpha\beta a^3}{3} + \frac{\beta^2 a^4}{12}\right)$$

2nd derivative $-(\alpha+\beta x)^2$ guaranteed negative so only one peak.

β and α solved for exactly.



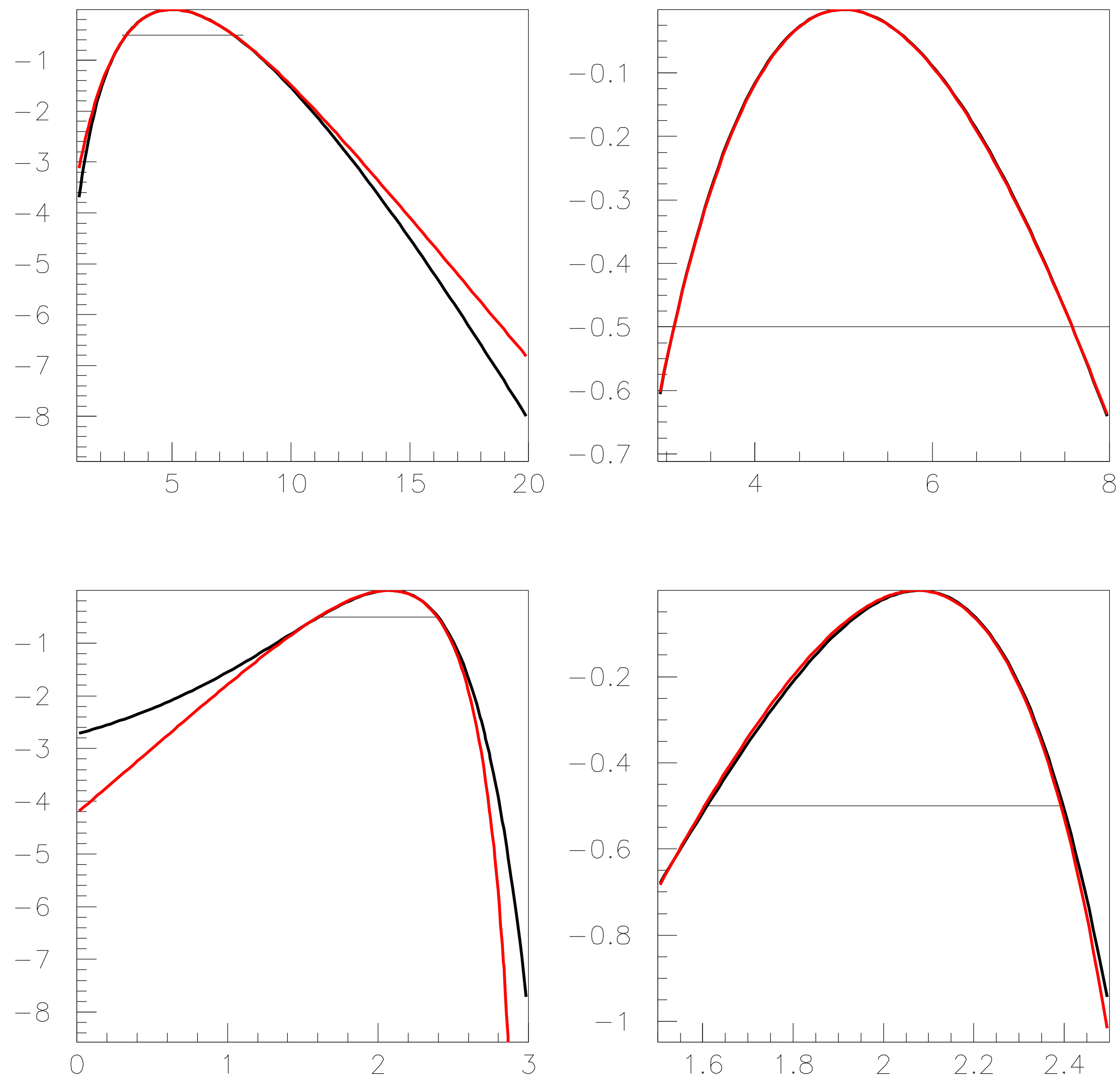
Agreement good in central region, does well outside for log Gauss, less impressive for Poisson

5: Logarithmic

$$\ln L = -\frac{1}{2}\left(\frac{\log(1+\gamma a)}{\log\beta}\right)^2$$

Continuous stretch/shrink of x axis

$\beta = \sigma_+/\sigma_-$ $\qquad$ $\gamma = \frac{\sigma_+ - \sigma_-}{\sigma_- \sigma_+}$



Pretty good. Problem when $1 + \gamma a$ goes negative.

6: Generalised Poisson

Poisson likelihood $N log a - a$, N integer.

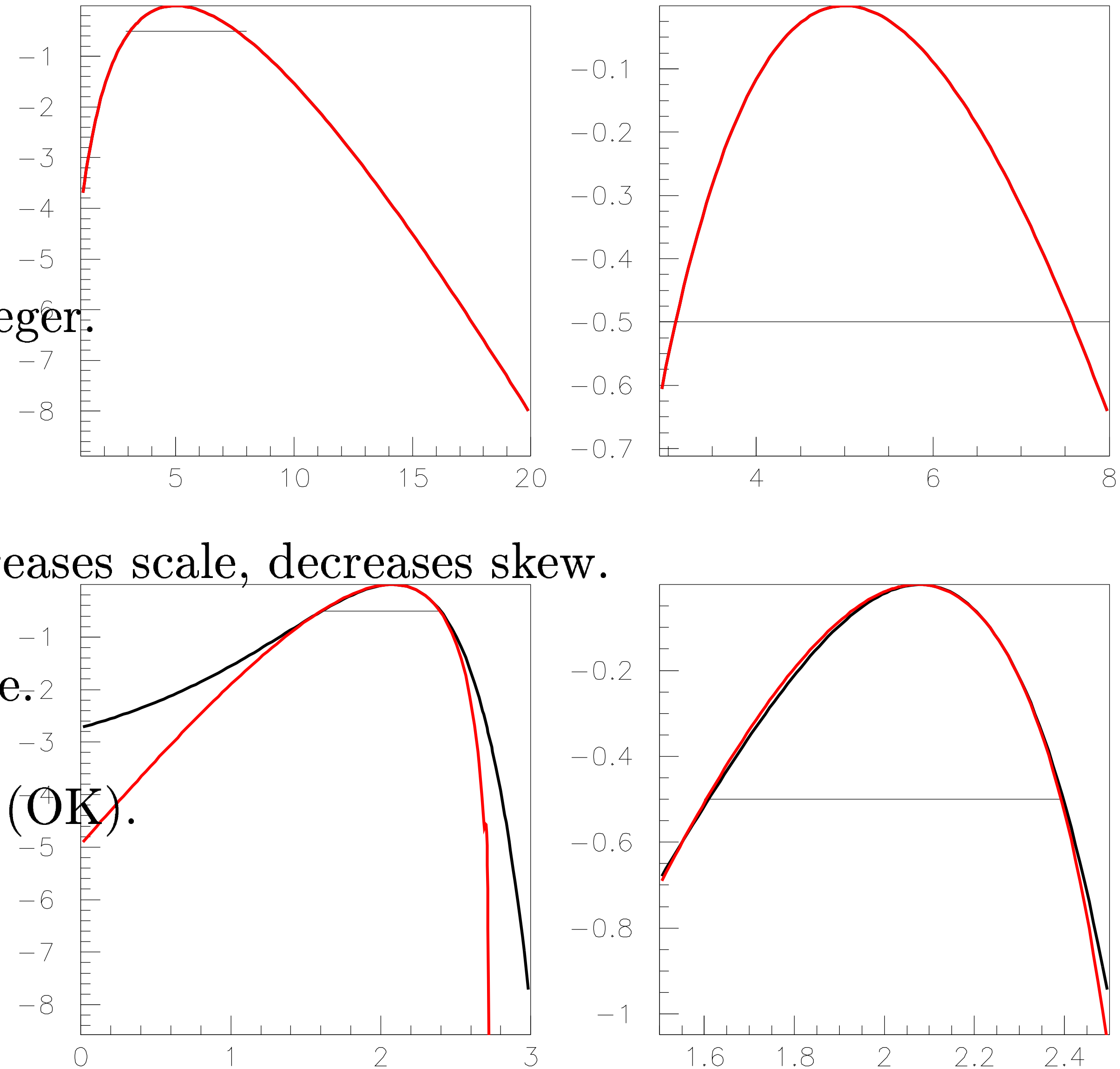
No problem in using this for real N.

Increasing N increases location, increases scale, decreases skew.

Introduce separate location and scale.

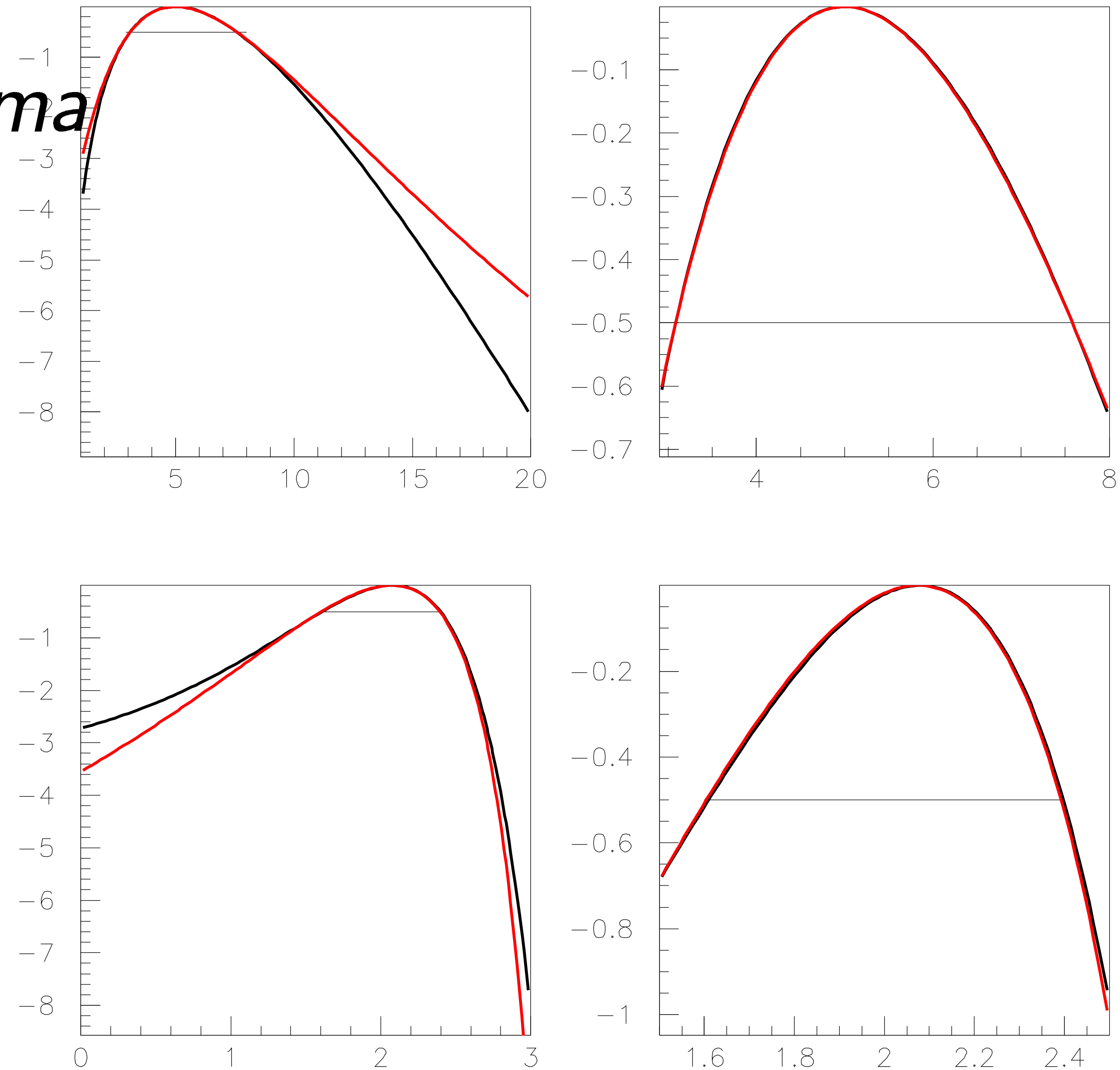
Parameters need numerical solution (OK).

$f(a) = -\alpha a + N ln(1 + \frac{\alpha a}{N})$



Perfect on Poisson, as expected. Not good on upper side of log Gaussian.

7: Gaussian - linear sigma



Motivation (Bartlett approx).
Gaussian with variable σ
$lnL(a) = -a^2/2\sigma(a)^2$

$$f(a) = -\frac{1}{2}\left(\frac{a^2}{\sigma_0+\sigma' a}\right)^2$$

$$\sigma_0 = \frac{2\sigma_+\sigma_-}{\sigma_++\sigma_-} \qquad \sigma' = \frac{\sigma_+-\sigma_-}{\sigma_++\sigma_-}$$

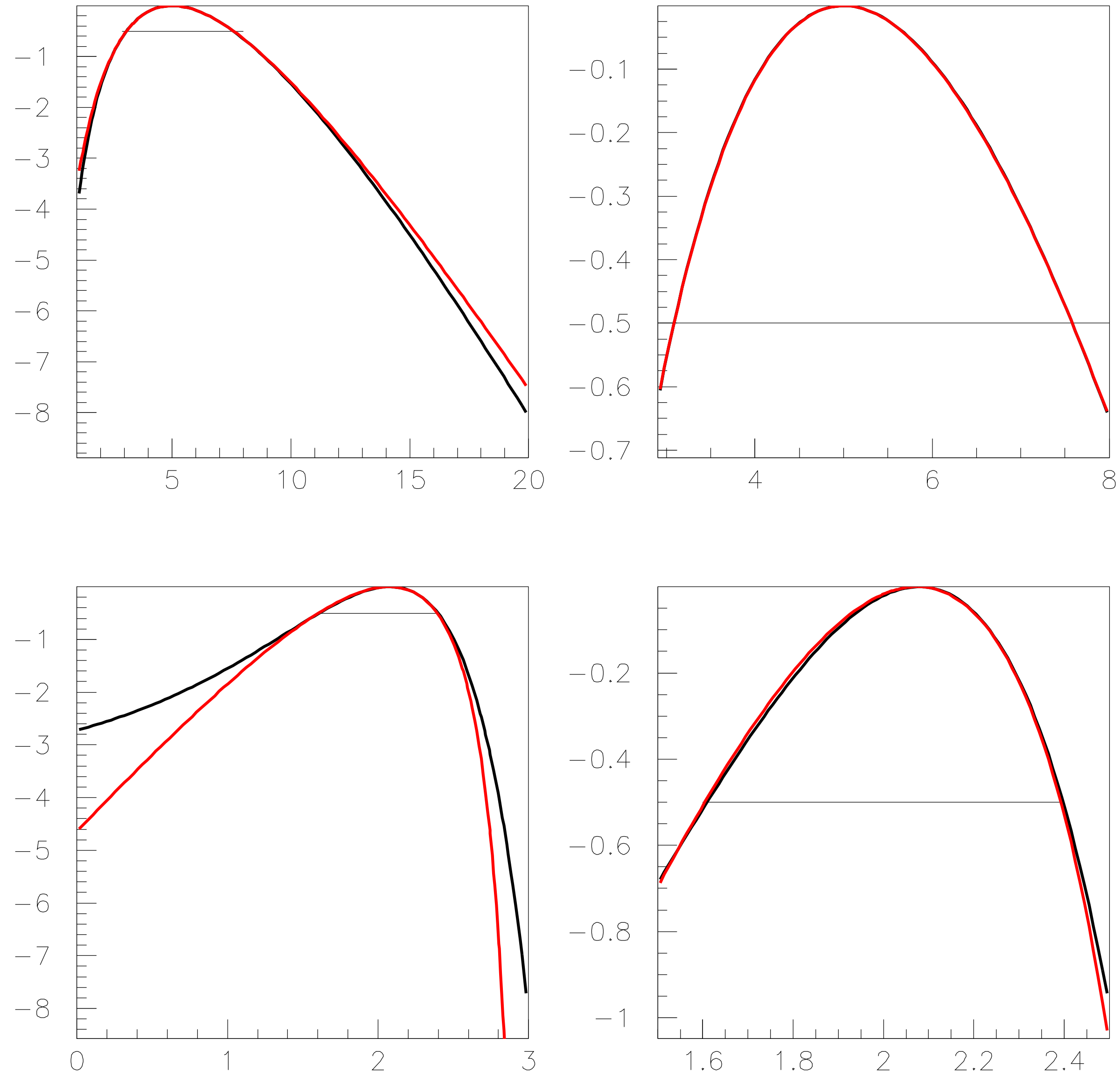
Excellent on log Gauss, pretty acceptable on Poisson

8: Gaussian - linear V

Same idea but change variance rather than σ

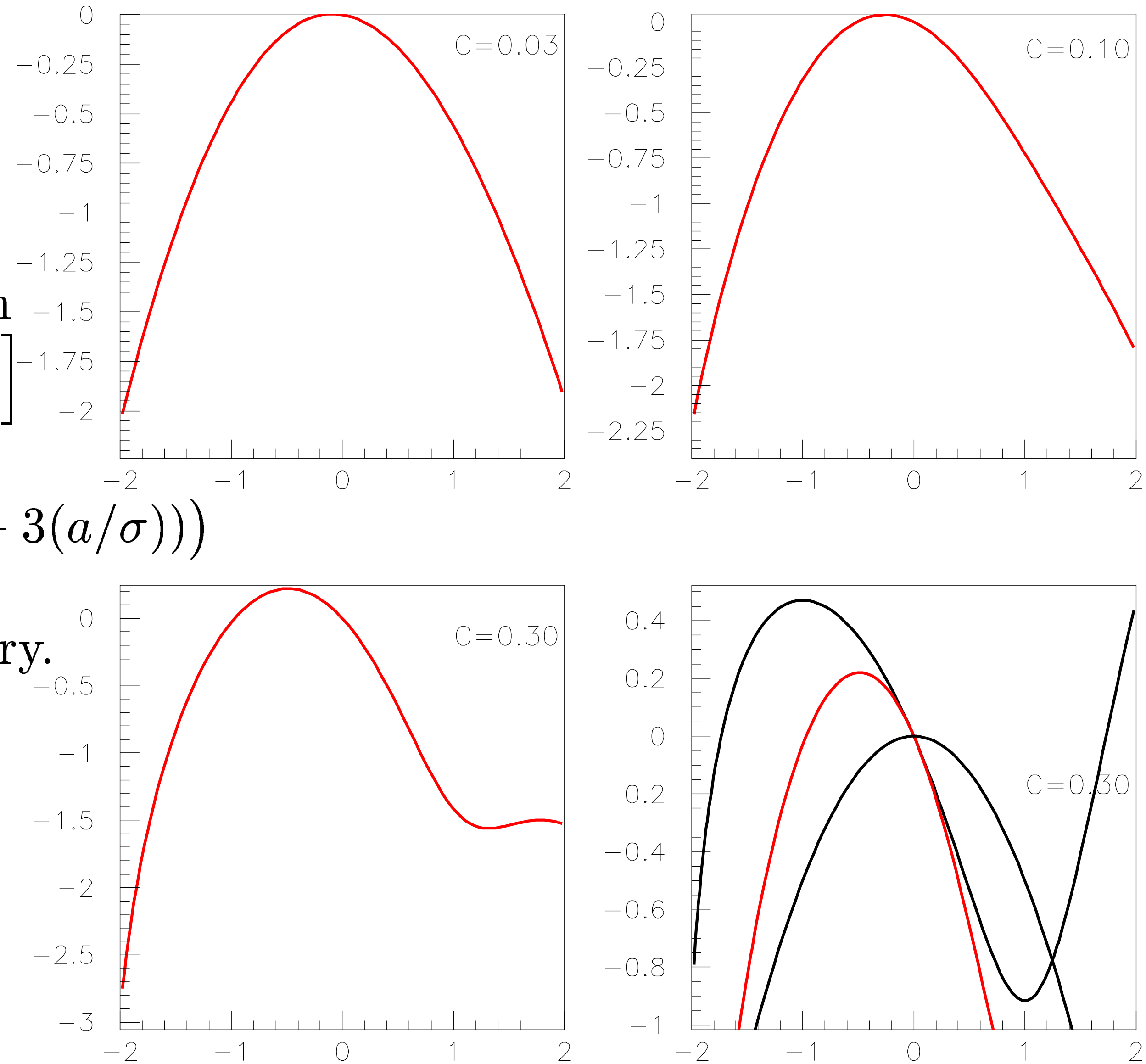
$f(a) = -\frac{1}{2}\frac{a^2}{V}$ with $V = V_0 + V'a$

$V_0 = \sigma_- \sigma_+ \qquad V' = \sigma_+ - \sigma_-$



Acceptable on log Gauss, Excellent on Poisson

Model 9: Edgeworth



Finite N samples have distribution

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \left[1 + \frac{\gamma_1}{6\sqrt{n}} H_3(x) + \frac{1}{n} (\ldots) \right]$$

$$f(a) = -\frac{1}{2}\frac{a^2}{\sigma^2} + ln(1 + C\left((a/\sigma)^3 - 3(a/\sigma)\right))$$

Doesn't work above 14% asymmetry.

Small C gives small asymmetry, but larger C just shifts peak.

Toy Experiment 1

Results from 2 identical Poisson experiments (mean of 10.0).
Might get, say, 9 and 15.

Correct procedure is to take $(9 + 15)/2 = 12.0$. But we only know that through privileged knowledge of experimental detail.

For general procedure have only results $9^{+3.34}_{-2.68}$ and $15^{+4.21}_{-3.55}$

1: Form the two likelihoods according to preferred model
2: Add to get combined likelihood
3: Read off result from maximum of combined likelihood
4: Read off errors from points where it falls by $\frac{1}{2}$

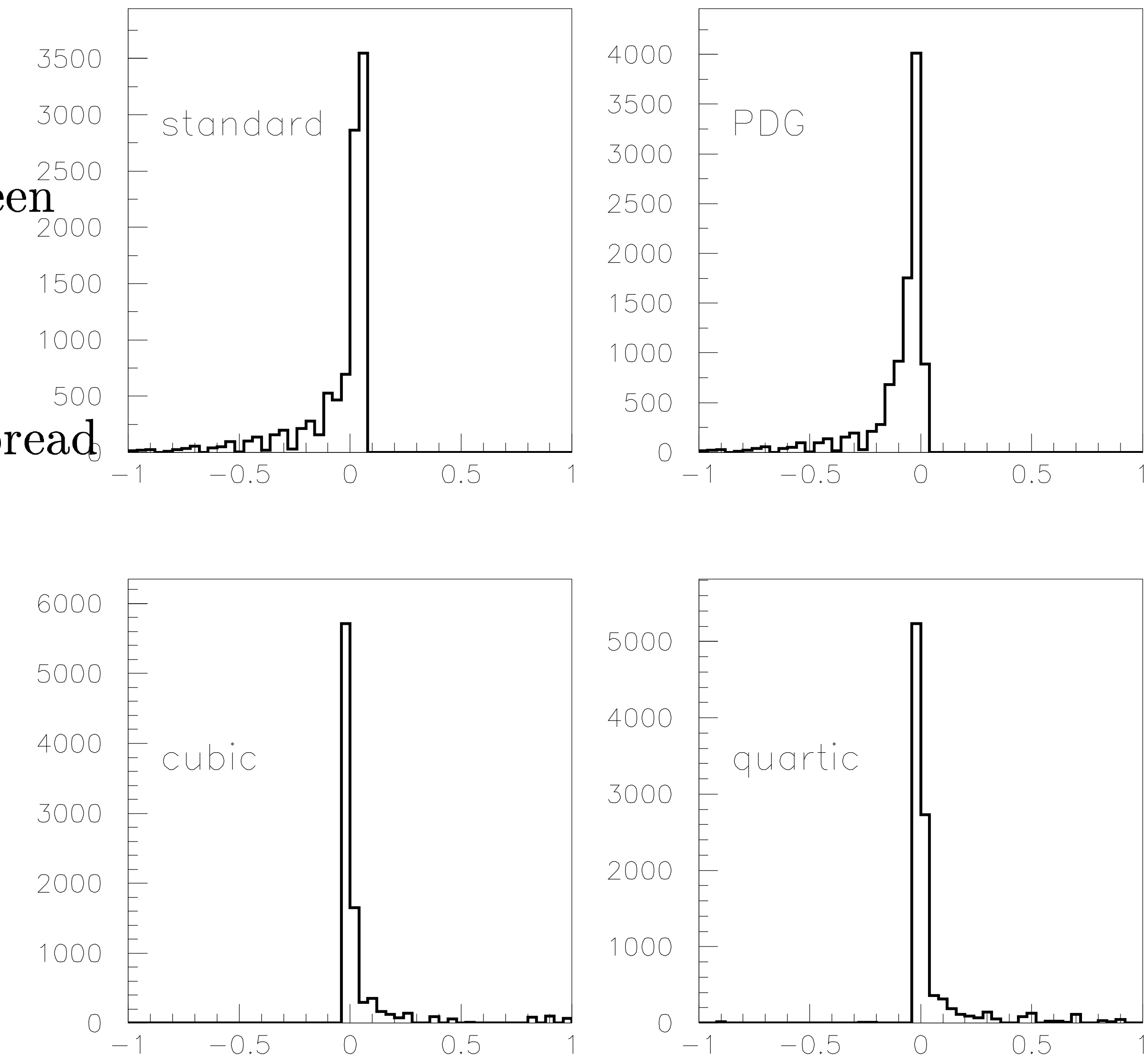
We do this 10,000 times

Toy Experiment 1

Histograms show deviations between correct (privileged) answer and general procedure for models 1-4

Methods 1 and 2 have bias and spread

Methods 3 and 4 do better



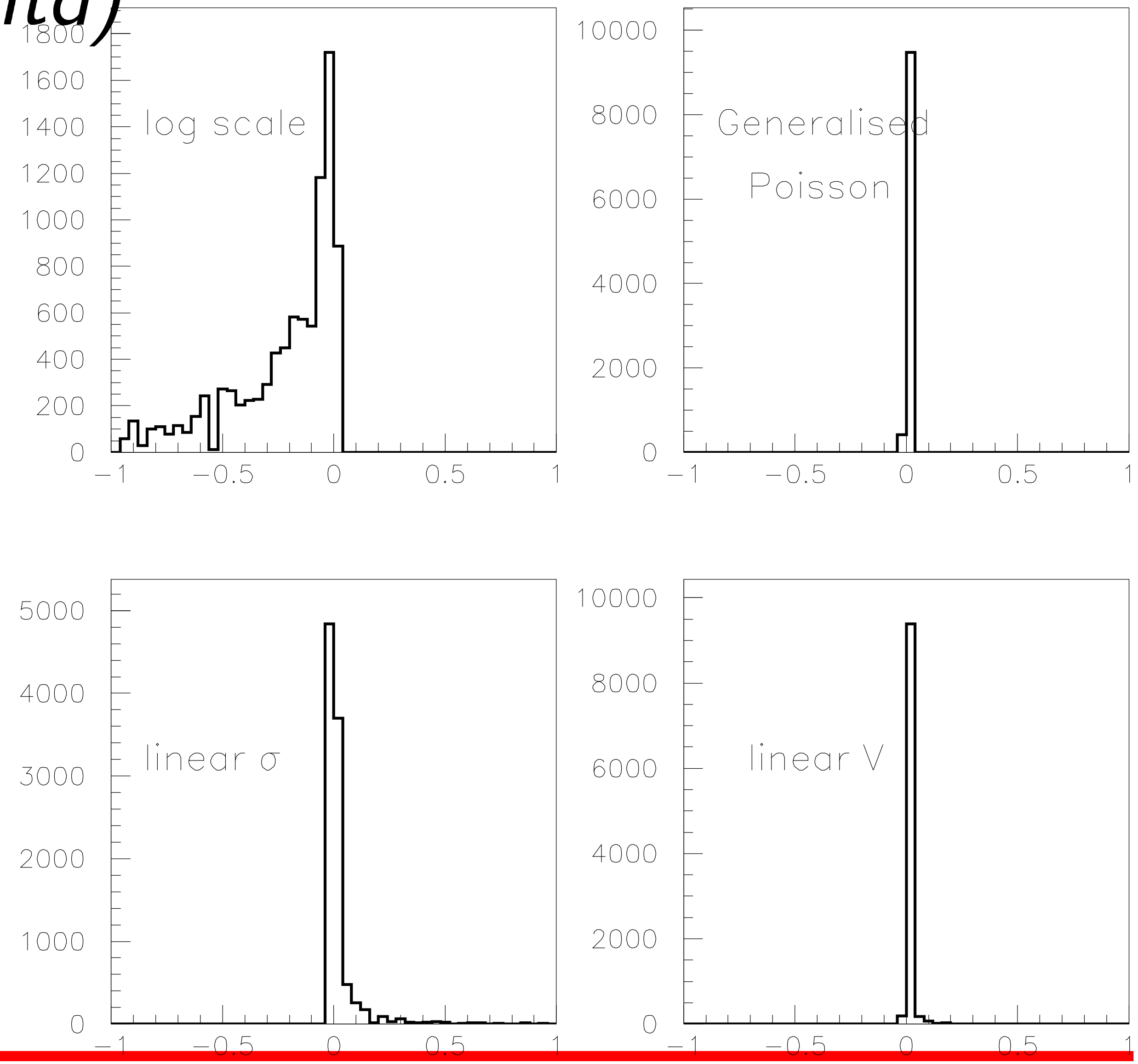
Toy Experiment 1 (contd)

Here are models 5-8

5 is awful (?)

7 is good

6 and 8 are brilliant



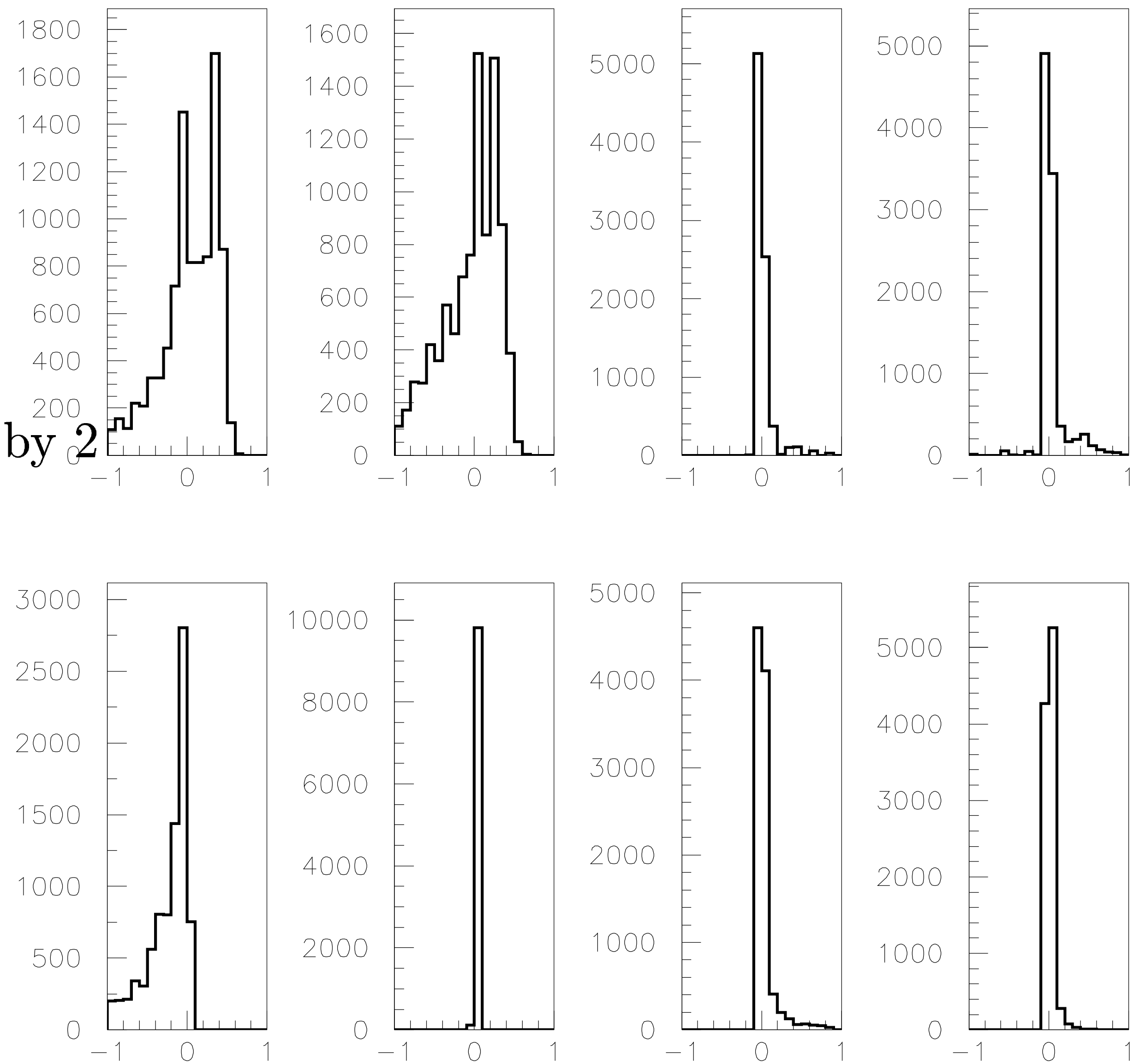
Toy Experiment 2

Again Poisson, one mean 10.0

Other ran for half the time
(so has mean 5.0) and is scaled up by 2

Similar pattern

Methods 1 and 2 even worse!



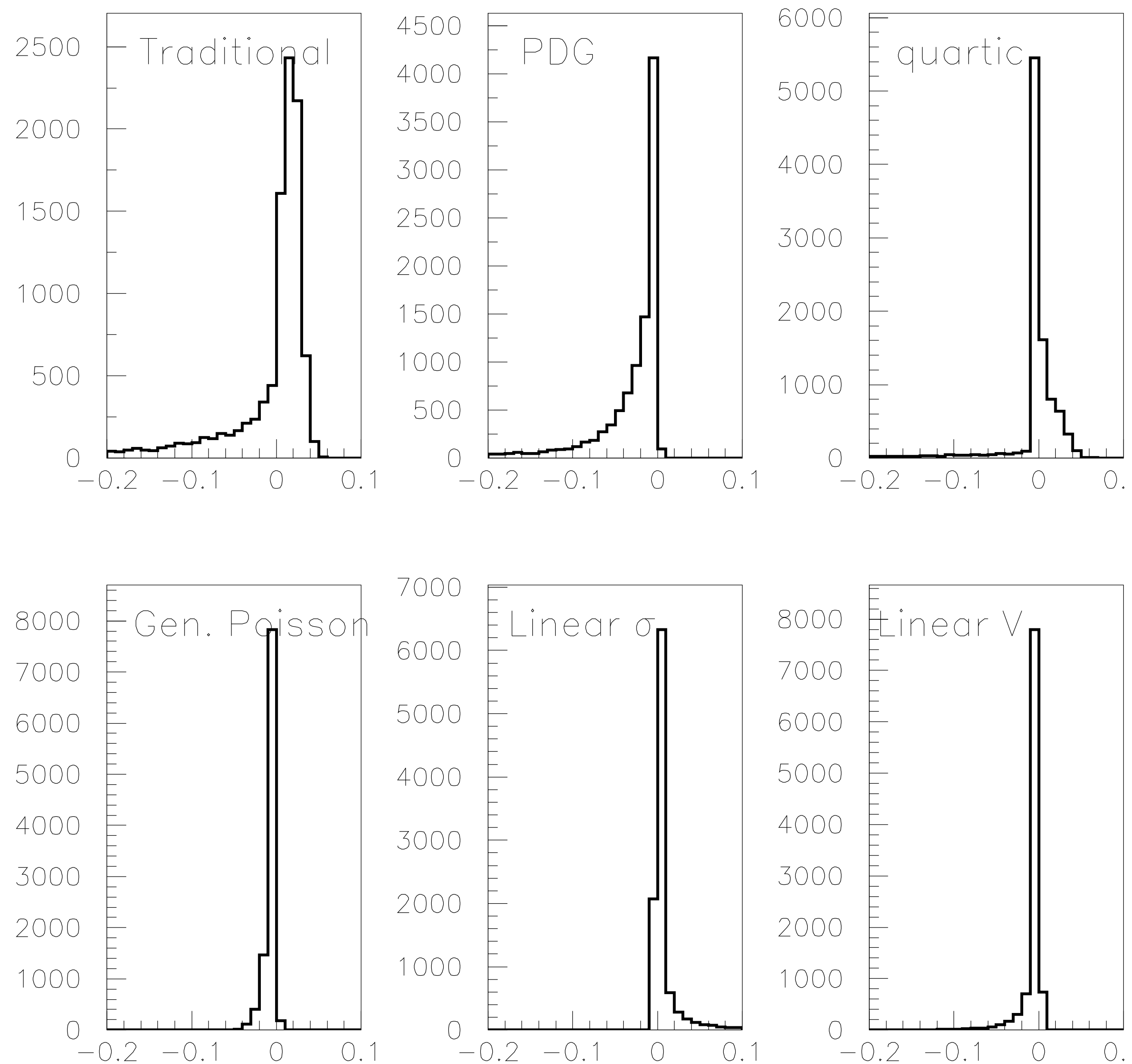
Toy Experiment 3

Two lifetime measurements, each based on 5 events

Correct (privileged) answer again simple average

$\Delta lnL = -\frac{1}{2}$ errors give
$\sigma_+ = 0.621\bar{t}$ $\quad\sigma_- = 0.340\bar{t}$

Generalised Poisson and linear variance Gaussian do best



Conclusions

More toy experiments needed, but provisionally:

- Can do better than traditional or PDG methods
- Generalised Poisson and Gaussian - linear V perform similarly well
- Gaussian - linear V parameters much easier to determine

Should be recommended

Comforting thought:
This is just the PDG method, but continuing the interpolation outside the $[-\sigma_-, \sigma_+]$ region, and using σ^2 rather than σ