Computation of Resistive Wakefields

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Resistive Wakefields

Longitudinal $m = 0$ wakefield
- Long Range approximation (Chao)
- Short Range approximation (Bane and Sands)
- Full treatment

Longitudinal $m > 0$ wakefields

Transverse wakefields

AC conductivity

Implementation

Scenario

Uniform circular pipe of radius b, conductivity σ

Interested in short range intrabunch fields

Assume thick pipe

Solve Maxwell's equations in vacuum and in metal pipe.

Decompose into angular modes $(cos(m\theta), m = 0, 1, 2...)$

Match boundary conditions.

Work in frequency space as differentiation $\rightarrow$ multiplication

The longitudinal wake for m=0

$$\tilde{E}_z(k) = \frac{2q}{b} \frac{1}{\frac{ikb}{2} - \left(\frac{\lambda}{k} + \frac{k}{\lambda}\right)\left(1 + \frac{i}{2\lambda b}\right)}$$

where

$$\lambda(k) = \sqrt{\frac{2\pi\sigma|k|}{c}}(i + sgn(k))$$

Introduce s_0, the *scaling length* (20μ for 1 cm Copper)

$$s_0 = \sqrt[3]{\frac{cb^2}{2\pi\sigma}} \qquad K = s_0 k \qquad s' = \frac{s}{s_0}$$

The simplest case: Chao's formula

In the long range (etc) limit $\tilde{E}_z = -\frac{2qk}{\lambda b}$

The Fourier Transform is well known to be $E_z(s) = \frac{q}{2\pi b}\sqrt{\frac{c}{\sigma}} s^{-\frac{3}{2}}$

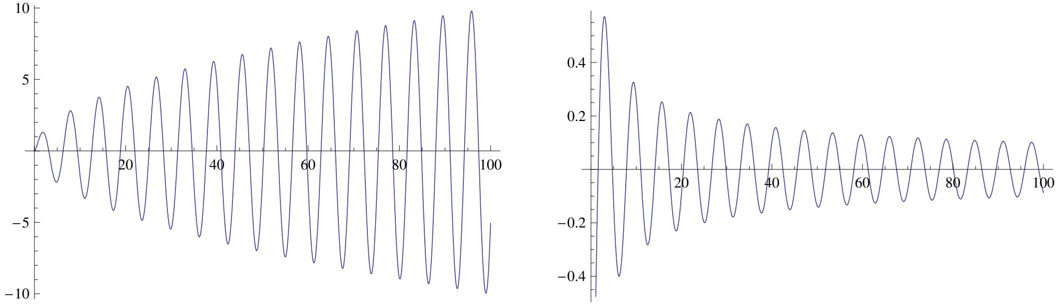
We do it the hard way by numerical integration of:

$$E_z(s) = \frac{1}{s_0 \pi} \int_0^\infty \Big(Re[f_{even}(K)] \cos(Ks') + Im[f_{odd}(K)] \sin(Ks') \Big) dK$$

$$f_{even}(K) = \frac{1}{2}\Big[f(K) + f(-K) \Big] = -\frac{q}{b^2}\sqrt{K}$$

$$f_{odd}(K) = \frac{1}{2}\Big[f(K) - f(-K) \Big] = \imath \frac{q}{b^2}\sqrt{K}$$

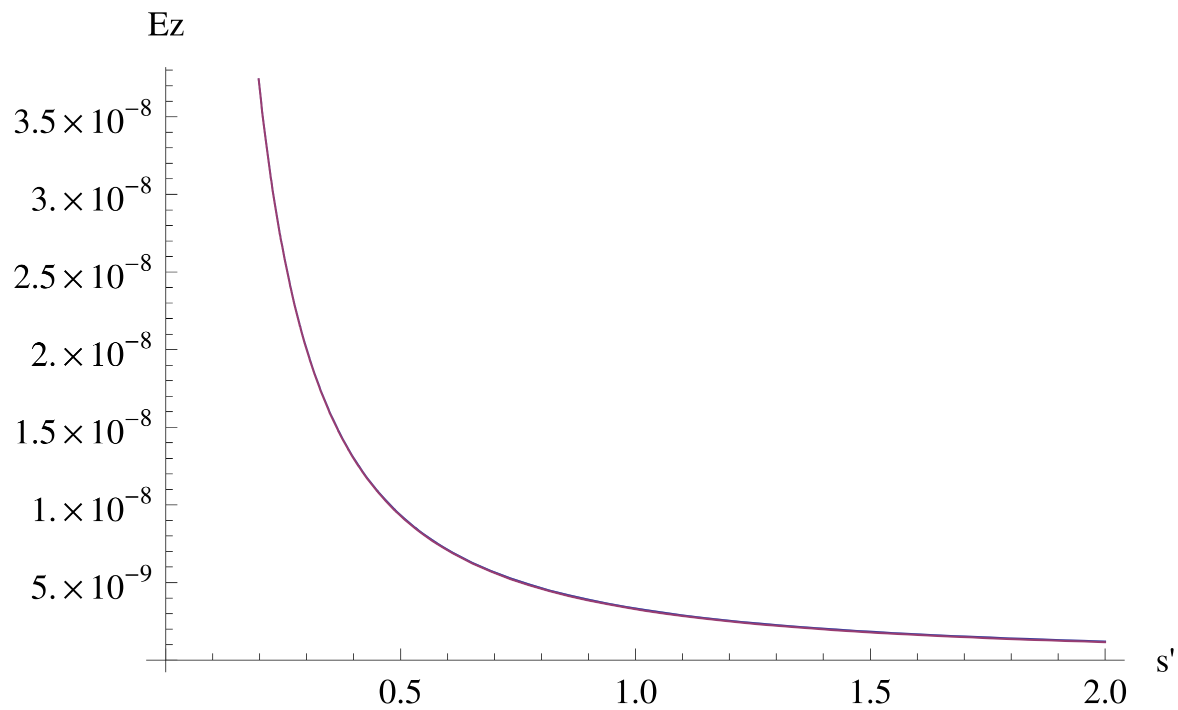
A problem and its solution



$\sqrt{K} sin K x$ oscillations increase. Numerical integration $\int_0^\infty dK$ hopeless

Solution:
First integrate analytically wrt x. Function becomes $-cos(Kx)\sqrt{K}/K$
Integrate numerically wrt K
Differentiate numerically wrt x

Long range wake



Actually has our results and the Chao formula superimposed.

A more accurate formula: Bane and Sands

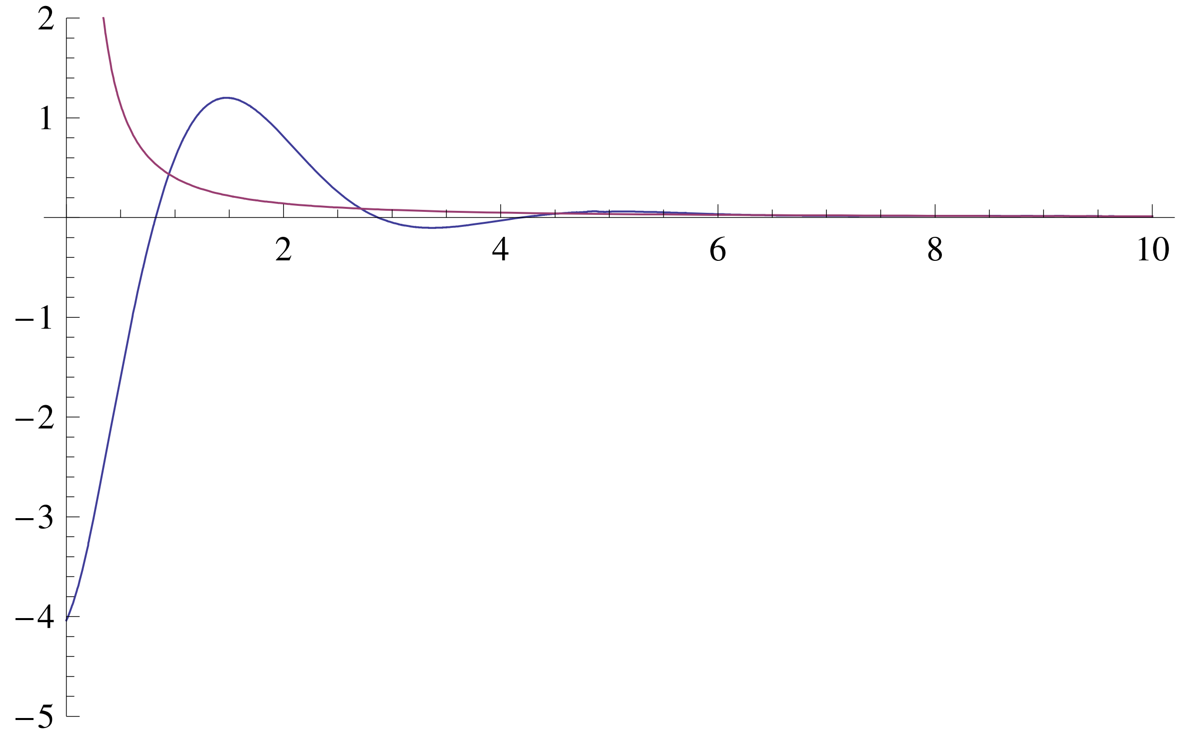
$$\tilde{E}_z = \frac{2q}{b} \frac{1}{\frac{ikb}{2} - \frac{\lambda}{k}}$$

Solution well known to be

$$E_z(s) = \frac{4qc}{\pi b^2} \left(\frac{e^{-s'}}{3} cos(\sqrt{3}s') - \frac{\sqrt{2}}{\pi} \int_0^\infty \frac{x^2 e^{-x^2 s'}}{x^6+8} dx \right)$$

We have

$$f_{even}(K) = -\frac{q}{b^2} \frac{\frac{2}{\sqrt{K}}}{\left(\frac{K}{2} - \frac{1}{\sqrt{K}}\right)^2 + \frac{1}{K}}, f_{odd}(K) = -\frac{q}{b^2} \frac{2\imath\left(\frac{K}{2} - \frac{1}{\sqrt{K}}\right)}{\left(\frac{K}{2} - \frac{1}{\sqrt{K}}\right)^2 + \frac{1}{K}}$$



Shows our results and Chao formula. Reproduces B&S

Wake is a function of three parameters (s, b and σ), but use of s_0 enables it to be written as a universal function $f(s')$, where $E_z(s, b) = \frac{q}{b^2} f(s/s_0)$.

The full formula

Full version, with $\xi = s_0^2/b^2$

$$f_{even}(K) = -\frac{8q}{b^2}\frac{\xi^2+\xi 2\sqrt{K}+4\frac{\sqrt{K}}{K}}{4\left[\xi\sqrt{K}-\frac{1}{K}\left(\xi+2\sqrt{K}\right)+K\right]^2+\left(\xi^2+\xi 2\sqrt{K}+4\frac{\sqrt{K}}{K}\right)^2}$$

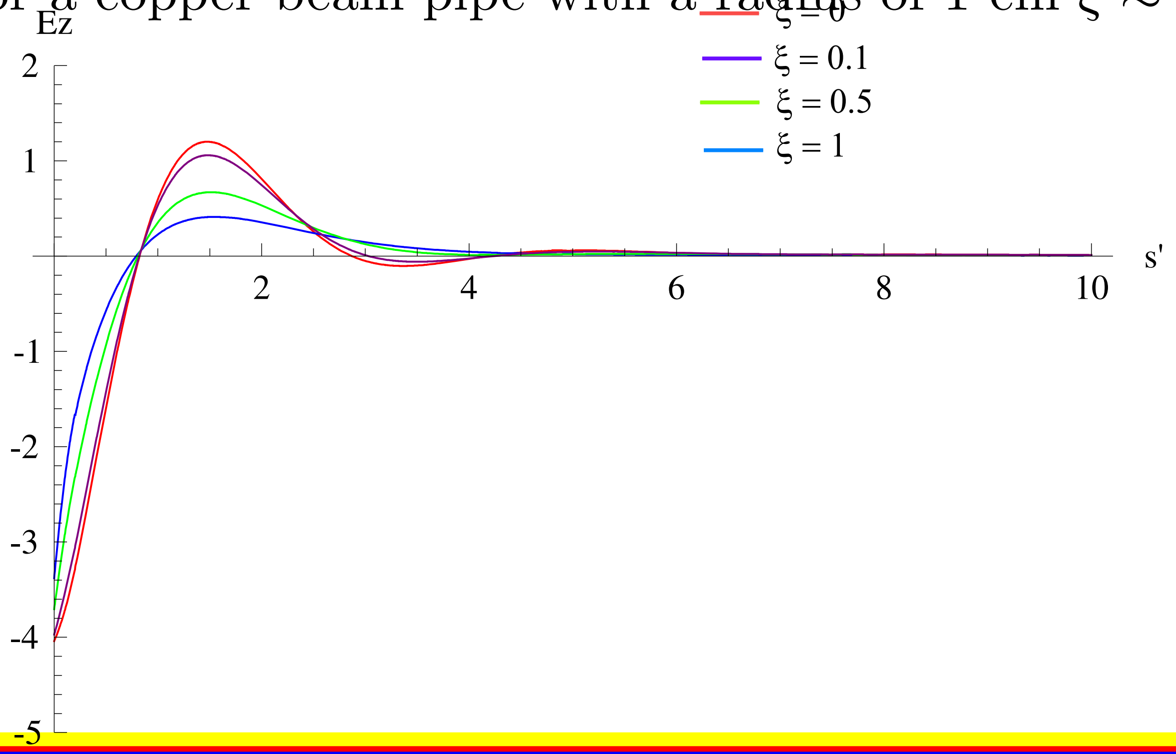
$$f_{odd}(K) = -\frac{8\imath q}{b^2}\frac{2\left[\xi\sqrt{K}-\frac{1}{K}\left(\xi+2\sqrt{K}\right)+K\right]}{4\left[\xi\sqrt{K}-\frac{1}{K}\left(\xi+2\sqrt{K}\right)+K\right]^2+\left(\xi^2+\xi 2\sqrt{K}+4\frac{\sqrt{K}}{K}\right)^2}$$

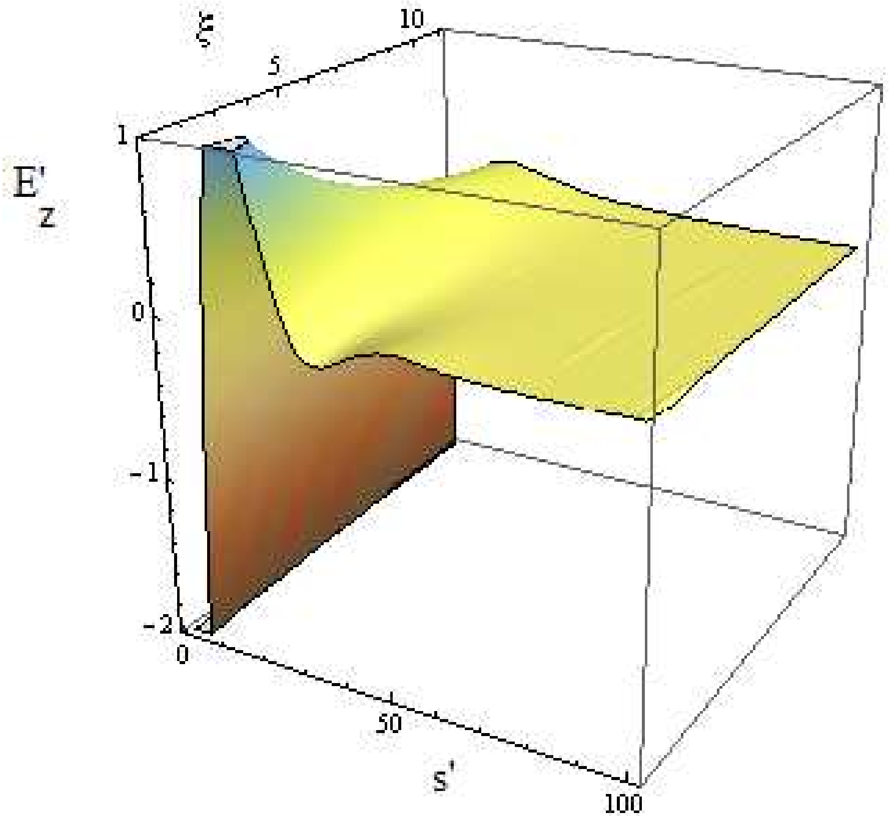
Although this is no longer a universal curve, it can still be expressed as a function of two variables (s' and ξ) rather than the full set of three. The B & S approximation corresponds to the function at $\xi = 0$.

Figure shows how the function changes for different values of ξ.

For ξ below about 0.1 the approximation is very good.

For a copper beam pipe with a radius of 1 cm $\xi \approx 0.000004$




ξ
10
5
1
E'z
0
-1
-2
0
50
s'
100

Longitudinal: Higher order modes

For higher modes, using the same technique for $m > 0$

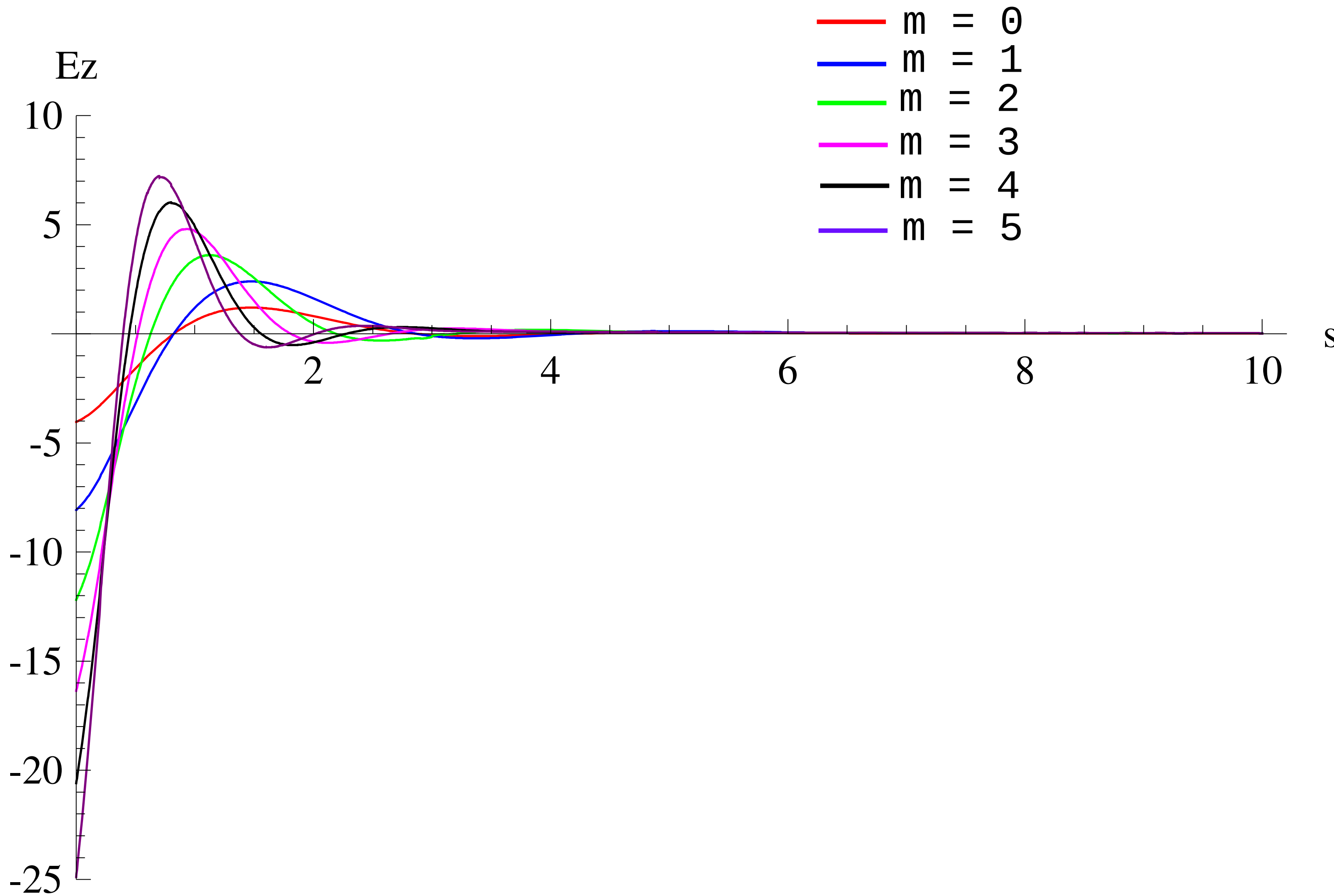
$$\tilde{E}_z^m = \frac{4}{b^{2m+1}} \frac{1}{\frac{\imath kb}{m+1} - \left(\frac{2k}{\lambda} + \frac{\lambda}{k}\right)\left(1 + \frac{\imath}{2\lambda b}\right) - \frac{\imath m}{kb}}$$

The equivalent of B & S is $\tilde{E}_z^m = \frac{4}{b^{2m+1}} \frac{1}{\frac{\imath kb}{m+1} - \frac{\lambda}{k}}$

This can be separated into odd and even parts,

$$f_{even} = -\frac{2}{b^{2m+1}} \frac{\frac{1}{\sqrt{K}}}{\left(\frac{K}{m+1} - \frac{1}{\sqrt{K}}\right)^2 + \frac{1}{K}} \quad f_{odd} = -\frac{2\imath}{b^{2m+1}} \frac{\left(\frac{K}{m+1} - \frac{1}{\sqrt{K}}\right)}{\left(\frac{K}{m+1} - \frac{1}{\sqrt{K}}\right)^2 + \frac{1}{K}}$$

Results

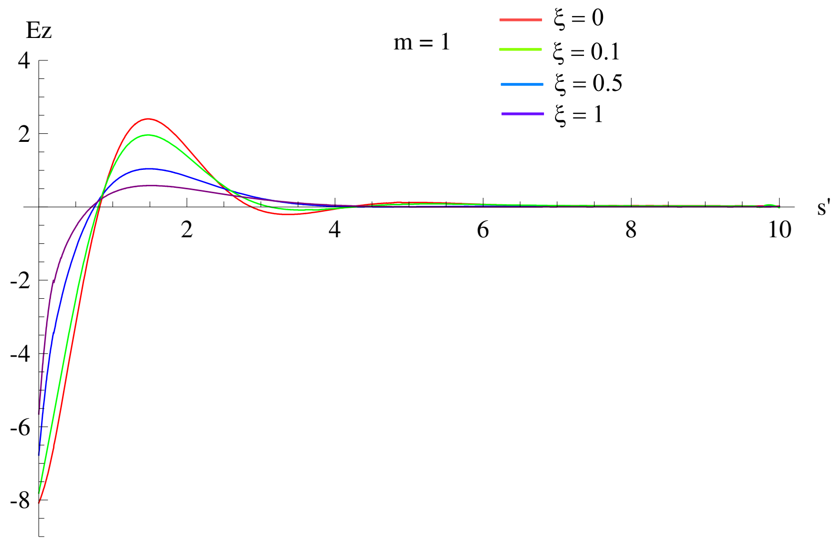


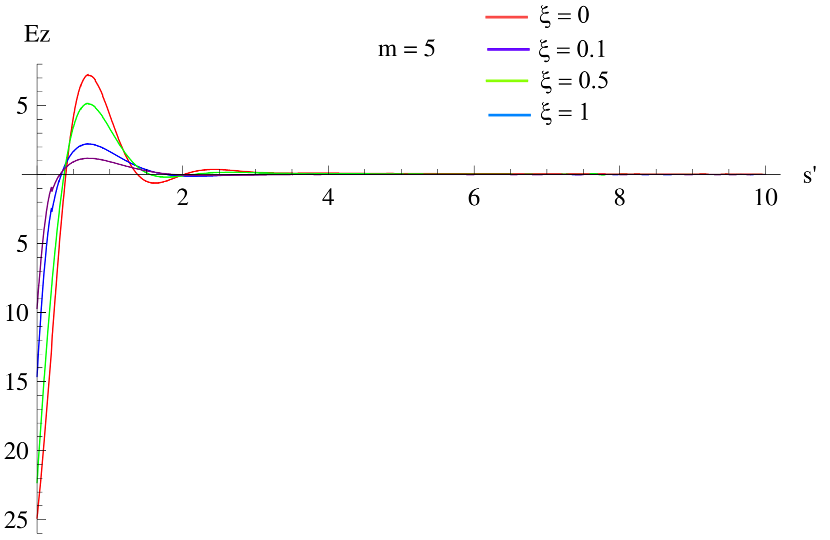
The full formula can be separated

$$f_{even}(K) = -\frac{8}{b^{2m+2}} \frac{\xi^2+\xi 2\sqrt{K}+2\frac{\sqrt{K}}{K}}{\left[\xi 2\sqrt{K}-\frac{1}{K}(\xi+2\sqrt{K})+2\left(\frac{K}{m+1}-\xi\frac{m}{K}\right)\right]^2+\left(\xi^2+\xi 2\sqrt{K}+2\frac{\sqrt{K}}{K}\right)^2}$$

$$f_{odd}(K) = -\frac{8\imath}{b^{2m+2}} \frac{\left[\xi 2\sqrt{K}-\frac{1}{K}(\xi+2\sqrt{K})+2\left(\frac{K}{m+1}-\xi\frac{m}{K}\right)\right]}{\left[\xi 2\sqrt{K}-\frac{1}{K}(\xi+2\sqrt{K})+2\left(\frac{K}{m+1}-\xi\frac{m}{K}\right)\right]^2+\left(\xi^2+\xi 2\sqrt{K}+2\frac{\sqrt{K}}{K}\right)^2}$$

Dependence on ξ for $m = 1$ and $m = 5$.

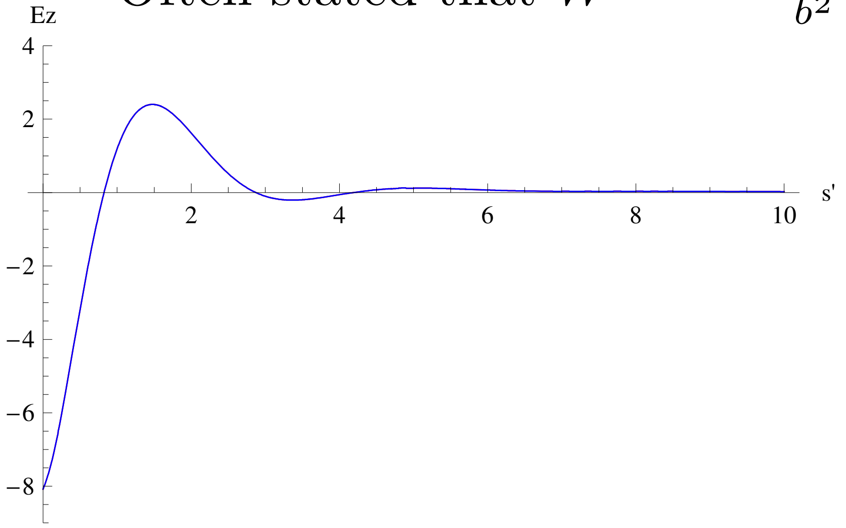


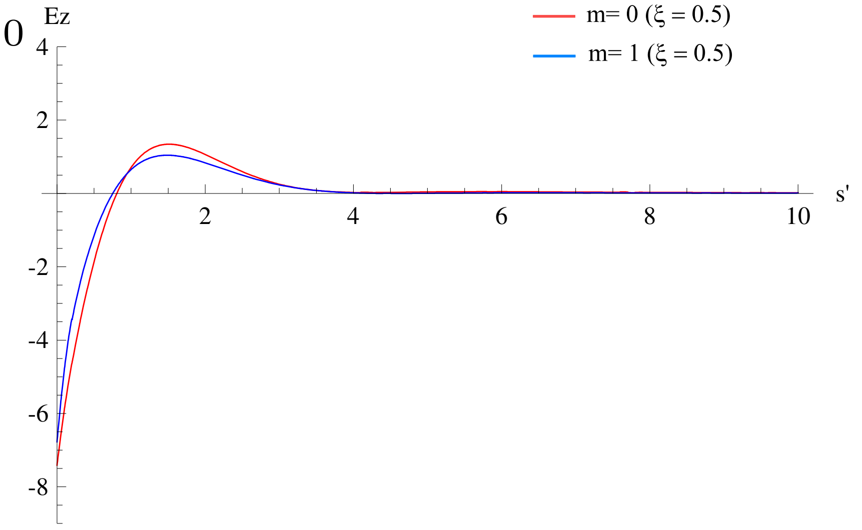


Dependence on ξ increases for higher modes but still looks ignorable for any sensible collimator.

Is m=1 proportional to m=0?

Often stated that $W^{m=1} = \frac{2}{b^2} W^{m=0}$





($m = 0$ shown doubled)

This is true for $\xi = 0$ but not in general.

And not for other m. Shapes different (See slide 13)

Transverse wakes

Transverse wake also a sum over angular modes

$$\vec{E}_T(r,\theta,s) = \sum_m r^{m-1}(\hat{r}cos(m\theta) - \hat{\theta}sin(m\theta))W_T^m(s)$$

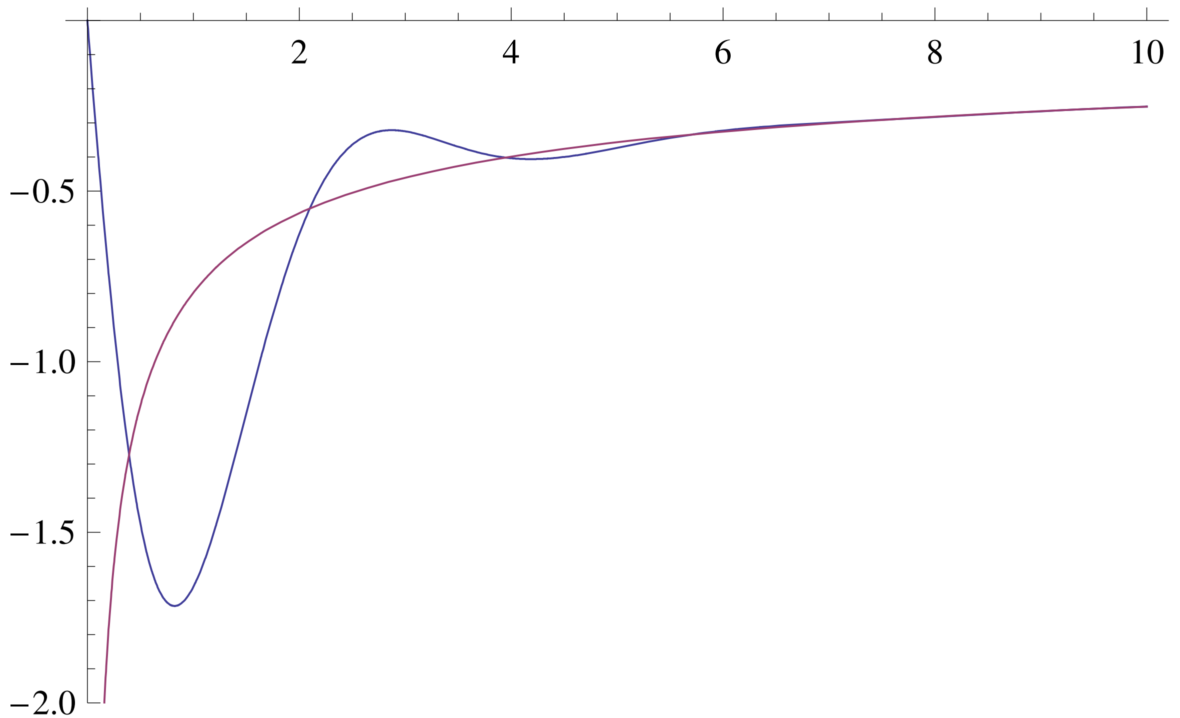
The Panofsky-Wenzel theorem $\nabla E_z = \frac{\partial \vec{E}_T}{\partial z}$

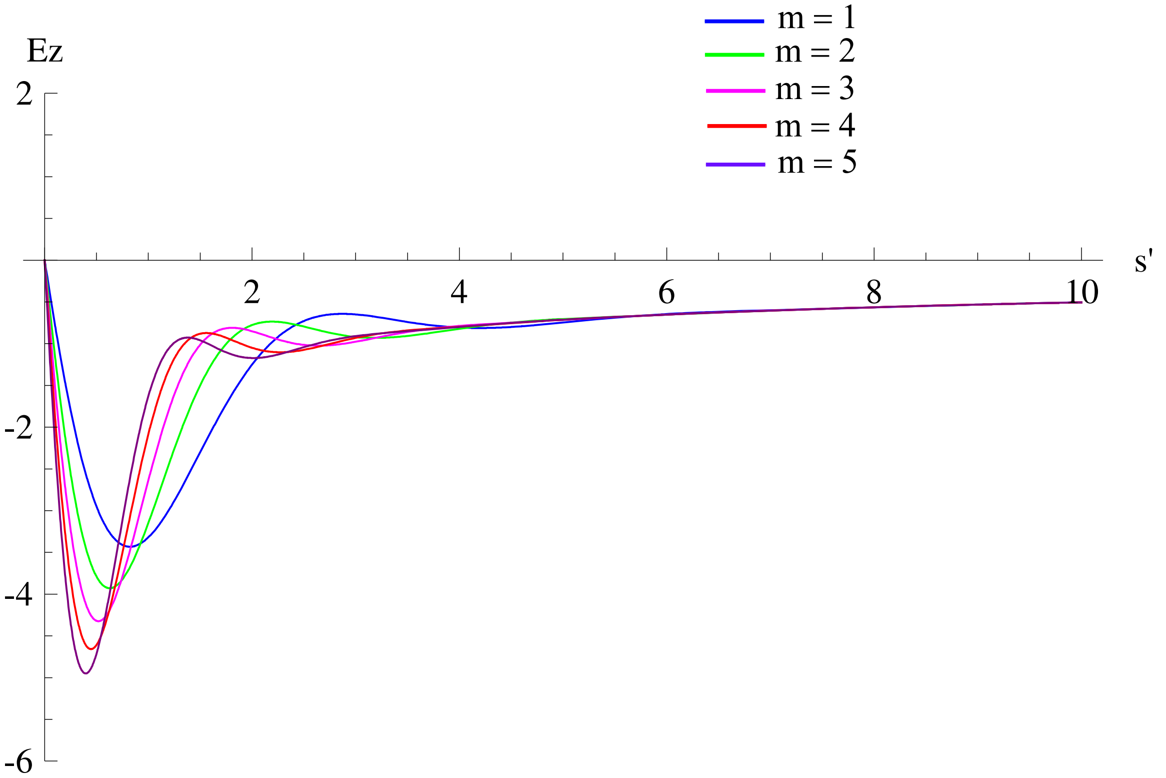
applies term by term giving $W_T^m(s) = \int_0^s E_z(x)dx$

Lucky we have that integral already evaluated!

Evaluate transverse wake

Using $\frac{2}{b^2}$ factor. Chao's formula also shown





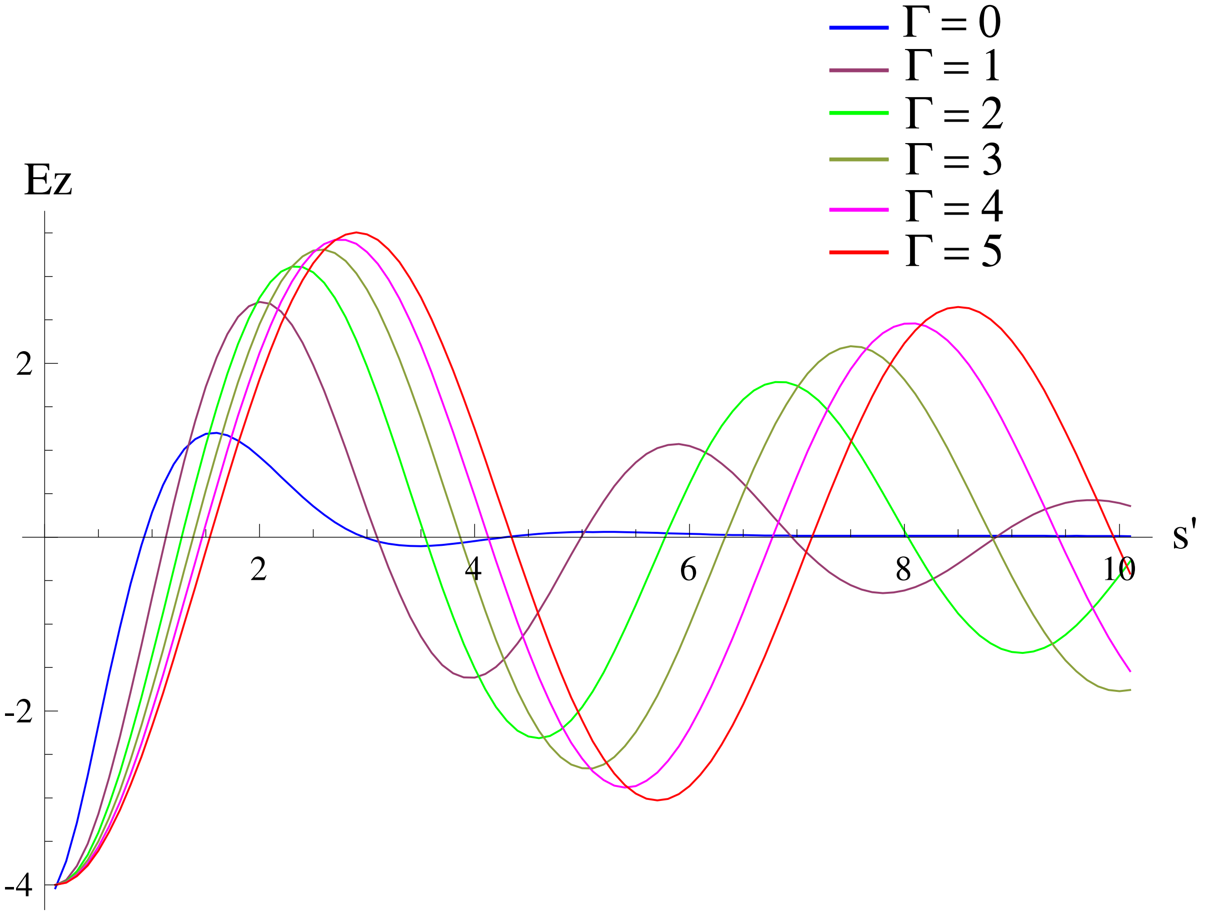
Transverse wakes - various modes with ξ=0

AC conductivity

$\tilde{\sigma} = \frac{\sigma}{1 - ikc\tau} = \frac{\sigma}{1 - iK\Gamma}$

Γ typically 1 at most. Usually much smaller.

Introduce into previous formulae - proceed as before



The $m = 0$ wake for various Γ

Implementation

Use Mathematica to integrate even and odd functions and generate table of values as function of s', ξ, Γ

Only 3 variables - and 2 of them don't vary (for a given collimator).

Write to file

`C++` object `collimatortable(file,Gamma,xi)` - portable

reads complete table and interpolates to get single table for s'

`double collimatortable::interpolate(double x)` returns the value

Implementation in MERLIN

Easy. Fits into existing structure introduced (see previous work) for geometric wakes.

Class `ResistivePotentials` inherits from `SpoilerWakePotentialsi`

Reads a set of tables from files when created and contains functions `Wtrans(z,m)` and `Wlong(z,m)` which each return a value from the tables (using parabolic interpolation), scaled by appropriate factors.

Only handles circular apertures

PLACET

Current version includes $m = 1$ transverse mode

Does include Anzsatz for rectangular collimators

$$y_{trailing} \rightarrow 0.822 \times y_{trailing} + 0.411 \times y_{leading}$$

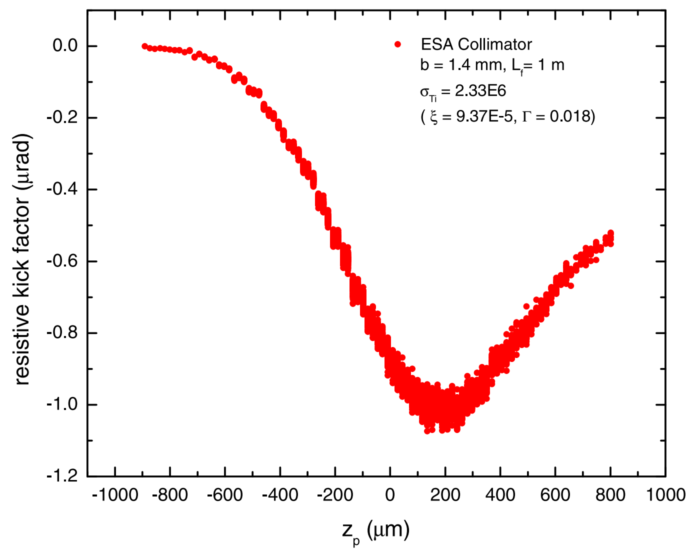
Implemented using `C++`

Examples

MERLIN used to evaluate resistive contributions to kick factors for ESA test collimators

Shown to be (much) less than geometric wakes (and $\leq$ measurement errors)

PLACET:



Future Work

Generate examples - CLIC and LHC

Make table (applies to ANY collimator) and program and documentation available

Extend to rectangular apertures